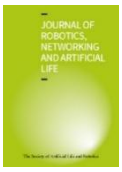
Journal of Robotics, Networking and Artificial Life
Vol. 11(2); December (2025), pp. 129-133
ISSN (Online): 2352-6386; ISSN (Print): 2405-9021
https://alife-robotics.org/jrnal.html



Research Article

Practical Linearization Control of Nonholonomic Wheeled Mobile Robots

Lixia Yan[1], Yingmin Jia[2, *]
[1]*Department of Automation, Tsinghua University, Beijing 100084, China*
[2]*The Seventh Research Division and the Center for Information and Control, School of Automation Science and Electrical Engineering, Beihang University (BUAA), Beijing 100191, China*
Email: *yanlixia@tsinghua.edu.cn*, *ymjia@buaa.edu.cn*
[*]Corresponding Author

ARTICLE INFO

Article History

Received 28 November 2024
Accepted 06 August 2025

Keywords

Practical linearization control
Practical stabilization
Nonholonomic wheeled mobile robots

ABSTRACT

Because of non-integrable constraints, we cannot regulate the nonholonomic systems towards arbitrary directions in the state space. The stabilization and tracking control laws developed for nonholonomic systems are generally mutual, i.e., the stabilization law is inapplicable for tracking uses and vice versa. The current work investigates the control problem of differentially-driven wheeled mobile robots and demonstrates an initial idea of a novel control design approach, named practical linearization control, for nonholonomic systems. We define an external dynamic oscillator and fuse it with robot states, followed by converting the nonholonomic robot model into a fully-actuated and linearizable one. Such fusion and conversion introduce a new control input without increasing the number of states to be regulated. Finally, we propose a continuous control law that can be used for both tracking and stabilization control tasks. It is, in Lyapunov's sense, proven that the tracking errors can be driven into an arbitrarily small ball enclosing the origin. Simulation results are carried out to validate the proposed control law.

1. Introduction

Typical differentially driven wheeled mobile robots (DWMRs) have no control input in the lateral direction, and feature nonholonomic constraint [1]. The existing controllers developed for such robots can be toughly divided into two types, namely, full-state [2] [3] [4] [5] and position-only control laws [6] [7], both of which focus on either stabilization or tracking purposes.

The stabilizers proposed for DWMRs are generally time-varying [2] or discontinuous [3] because the kinematic model of DWMRs does not satisfy the Brockett's necessary condition. In [2], the designed visual control law is developed by fusing states with an oscillating time-varying function, achieving pose stabilization with the unknown camera focal length. To ensure exponential convergence of pose errors, the discontinuous control law reported in [3] has combined the σ-process and switching mechanism. In [4], a tracking control law at the acceleration level is designed and makes the position and attitude errors converge to zero uniformly asymptotically. A novel control scheme allowing for simultaneous stabilization and tracking is proposed in [5]. Via choosing a new position coordinate in front of the body center, the control schemes in [6] [7] convert the derivative of the new position into a form that can be feedback linearized, making linear methods applicable to the control design. However, normal feedback linearization control laws like those in [6] [7] leave the orientation uncontrolled and do not consider the zero dynamics associated with orientation. As a result, how orientation behaves in the steady state remains unknown. Therefore, it is of great significance to investigate the feedback linearization control scheme that considers all states of DWMRs.

Motivated by the literature review above, we generalize the results in our previous work [5] and propose an initial idea of the practical linearization control scheme for DWMRs. More precisely, the DWMR pose states are fused with external oscillators whose derivatives introduce a new virtual control input. Within such a trick, the kinematic model of DWMRs can be converted into a new feedback linearization form. Then, we design a linear control law in terms of feedback linearization and proportional control, and prove that the pose error would be driven converging into an arbitrarily small ball enclosing the origin, no matter whether the reference pose signals are time-invariant or time-varying.

The rest of the work is organized as follows. The problem formulation and control design are included in Section 2. Section 3 presents the numerical simulation results. A brief conclusion is made in Section 4.

2. Main results

2.1. Problem formulation

The motion of a typical NWMR can be controlled by regulating the velocities of two wheels. Let (u_r, u_l) be velocities of left and right wheels, the kinematic model of a DWMR can be described by [5].

$$\dot{x} = \frac{u_r + u_l}{2}\cos\theta, \dot{y} = \frac{u_r + u_l}{2}\sin\theta, \dot{\theta} = \frac{u_r - u_l}{d} \tag{1}$$

where $[x, y]^T$ stands for the position, θ denotes the orientation, and d represents the distance between two wheels. Define $u = 0.5(u_r + u_l), r = (u_r - u_l)/d$ and convert the model (1) into

$$\dot{x} = u\cos\theta, \dot{y} = u\sin\theta, \dot{\theta} = r \tag{2}$$

Suppose that the reference pose $[x_d, y_d, \theta_d]^T$ obeys the kinematics as (2) and meets the following assumptions,

Assumption 1. The derivatives of reference pose with respect to time are bounded, i.e., $\dot{x}_d, \dot{y}_d, \dot{\theta}_d \in L_\infty$.

Assumption 2. One can always find a constant $\gamma \in \mathbb{R}_{>0}$ so that

$$\left\| [x_d, y_d, \theta_d] \right\|_2 \le \gamma \tag{3}$$

The control objective can then be stated as: find a control law for (u, r) within Assumptions 1-2 such that

$$\lim_{t\to\infty} \left\| [x, y, \theta] - [x_d, y_d, \theta_d] \right\| \le \delta \tag{4}$$

where δ denotes a small positive number.

The reference pose signal can be time-varying or time-invariant under Assumptions 1-2. This means that we are going to find a control law that can simultaneously deal with the tracking and stabilization problem of DWMRs.

Remark 1. In view of the kinematic model (2), one could apply the normal feedback linearization approach and define new coordinates below,

$$\begin{bmatrix} \bar{x}_1 \\ \bar{y}_1 \end{bmatrix} = \begin{bmatrix} x + l\cos\theta \\ y + l\sin\theta \end{bmatrix} \to \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{y}}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -l\sin\theta \\ \sin\theta & l\cos\theta \end{bmatrix}}_{M} \begin{bmatrix} u \\ r \end{bmatrix} \tag{5}$$

where l is a constant. Obviously, the derivatives of the new coordinates in (5) demonstrate that the $[\dot{\bar{x}}_1, \dot{\bar{y}}_1]$-dynamics is feedback linearizable as the matrix M is invertible. However, such a trick does not consider the motion of orientation. How the orientation angle θ behaves is unclear and has been rarely addressed.

2.2. Control design

Define the new coordinates in the body-fixed frame,

$$\theta_1 = \theta$$
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{6}$$

and its derivative can be calculated as,

$$\dot{\theta}_1 = r, \dot{x}_1 = u + ry_1, \dot{y}_1 = -rx_1 \tag{7}$$

The core of practical linearization is to fuse states with an external dynamic oscillator and introduce an additional control input. Choose an intermediate variable $\phi \in \mathbb{R}$ and choose a small positive number ε, we define,

$$\begin{cases} z_1 = \theta_1 - \varepsilon\cos\phi \\ z_2 = x_1 - \varepsilon\sin\phi \\ z_3 = 2y_1 + (z_1 + \varepsilon\cos\phi)(z_2 + \varepsilon\sin\phi) \end{cases} \tag{8}$$

where the combination $(\varepsilon\cos\phi, \varepsilon\sin\phi)$ is the dynamic oscillator. Differentiating (8) with respect to time yields

$$\begin{cases} \dot{z}_1 = r + \varepsilon\dot{\phi}\sin\phi \\ \dot{z}_2 = u + ry_1 - \varepsilon\dot{\phi}\cos\phi \\ \dot{z}_3 = \varepsilon^2\dot{\phi} - \dot{z}_1(z_2 + 2\varepsilon\sin\phi) + \dot{z}_2(z_1 + 2\varepsilon\cos\phi) \end{cases} \tag{9}$$

Obviously, the new dynamics (9) is fully actuated if we adopt $\varepsilon^2\dot{\phi}$ to regulate z_3. Define new control inputs by

$$w_1 \triangleq \dot{z}_1, w_2 \triangleq \dot{z}_2, w_3 \triangleq \varepsilon^2\dot{\phi} \tag{10}$$

Convert (9) into

$$[\dot{z}_1, \dot{z}_2, \dot{z}_3]^T = A[w_1, w_2, w_3]^T \tag{11}$$

where the matrix A is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -z_2 - 2\varepsilon\sin\phi & z_1 + 2\varepsilon\cos\phi & 1 \end{bmatrix} \tag{12}$$

As can be seen, the newly obtained model (11) can be feedback linearized as the matrix A is invertible by the fact $|A| = 1$. Following the same routine as (6) and (8), we define,

$$\begin{cases} x_{1d} = x_d\cos\theta_d + y_d\sin\theta_d \\ y_{1d} = -x_d\sin\theta_d + y_d\cos\theta_d \\ \theta_{1d} = \theta_d \\ \dot{x}_{1d} = u_d + r_d y_{1d}, \dot{y}_{1d} = -r_d x_{1d}, \dot{\theta}_{1d} = r_d \end{cases} \tag{13}$$

Let

$$z_{1d}=\theta_{1d}, z_{2d}=x_{1d}, z_{3d}=2y_{1d}+z_{1d}z_{2d} \tag{14}$$

and define tracking errors,

$$e_1=z_1-z_{1d}, e_2=z_2-z_{2d}, e_3=z_3-z_{3d} \tag{15}$$

The lemma below reveals that stabilizing errors defined in (15) is equivalent to achieving the control objective (4).

Lemma 1. If $e_1 \to 0, e_2 \to 0, e_3 \to 0$ as $t \to +\infty$, we can find a positive constant δ so that (4) establishes.

Proof. According to the statements in the lemma and new states given by (8), one has,

$$\begin{aligned} &\theta_1-\theta_{1d}=\theta-\theta_d \to \varepsilon\cos\phi \\ &x_1-x_{1d} \to \varepsilon\sin\phi \\ &y_1-y_{1d} \to -\varepsilon\frac{z_{1d}\sin\phi+z_{2d}\cos\phi+\varepsilon\cos\phi\sin\phi}{2} \end{aligned} \tag{16}$$

For the term y_1-y_{1d}, it is clear that

$$\lim_{t\to\infty}\|y_1-y_{1d}\| \le 0.5\varepsilon\left(\sqrt{z_{1d}^2+z_{2d}^2}+\varepsilon\right) \tag{17}$$

which, together with the relationship $z_{2d}^2 \le x_d^2+y_d^2$ and Assumption 2, leads to

$$\lim_{t\to\infty}\|y_1-y_{1d}\| \le 0.5\varepsilon(\gamma+\varepsilon) \tag{18}$$

Henceforth,

$$\left\|[x_1-x_{1d}, y_1-y_{1d}]\right\| \le 0.5\varepsilon(\gamma+\varepsilon+2) \tag{19}$$

Define a rotation matrix $S(*)=\begin{bmatrix}\cos* & \sin* \\ -\sin* & \cos*\end{bmatrix}$, we use the sum-to-product formula and obtain

$$\begin{aligned} \begin{bmatrix} x_1-x_{1d} \\ y_1-y_{1d}\end{bmatrix} &= S(\theta)\begin{bmatrix}x\\y\end{bmatrix}-S(\theta_d)\begin{bmatrix}x_d\\y_d\end{bmatrix} \\ &=\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x-x_d\\y-y_d\end{bmatrix} \\ &\quad+\left[S(\theta)-S(\theta_d)\right]\begin{bmatrix}x_d\\y_d\end{bmatrix} \end{aligned} \tag{20}$$

where

$$\begin{aligned} S(\theta)-S(\theta_d) &= \begin{bmatrix}\cos\theta-\cos\theta_d & \sin\theta-\sin\theta_d \\ -\sin\theta+\sin\theta_d & \cos\theta-\cos\theta_d\end{bmatrix} \\ &=2\sin\frac{\theta-\theta_d}{2}\begin{bmatrix}-\sin\frac{\theta+\theta_d}{2} & \cos\frac{\theta+\theta_d}{2} \\ -\cos\frac{\theta+\theta_d}{2} & -\sin\frac{\theta+\theta_d}{2}\end{bmatrix} \end{aligned}$$

Therefore,

$$\left\|\left[S(\theta)-S(\theta_d)\right]\begin{bmatrix}x_d\\y_d\end{bmatrix}\right\| \le \|\theta-\theta_d\|\left\|[x_d, y_d]\right\| \tag{21}$$

By Assumption 2, $\left\|[x_d, y_d]\right\| \le \left\|[x_d, y_d, \theta_d]\right\|$ and (20), one yields,

$$\lim_{t\to+\infty}\left\|[x-x_d, y-y_d]\right\| \le 0.5\varepsilon(3\gamma+\varepsilon+2) \tag{22}$$

Define the ultimate bound δ by,

$$\delta=0.5\varepsilon(3\gamma+3\varepsilon+2) \tag{23}$$

The claims in the lemma are hence established. □

Since the new model is feedback linearizable, the control design becomes straightforward. We choose a positive matrix $K=\operatorname{diag}\{k_1, k_2, k_3\}$ and design,

$$\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}=A^{-1}\left\{\begin{bmatrix}\dot z_{1d}\\ \dot z_{2d}\\ \dot z_{3d}\end{bmatrix}-K\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix}\right\} \tag{24}$$

where the operator 'diag' stands for diagonalizing the vector. Using (9)(10), and (24), we can recover the original control inputs via,

$$\dot\phi=\frac{w_3}{\varepsilon^2}, r=w_1-\varepsilon\dot\phi\sin\phi, u=w_2-ry_1+\varepsilon\dot\phi\cos\phi \tag{25}$$

and the wheel velocities can be calculated as,

$$u_r=u+0.5dr, u_l=u-0.5dr \tag{26}$$

The theorem below summarizes the main results of the current work.

Theorem 1. If Assumptions 1-2 hold, the application of (24)(25) on the NWMR model (1) would achieve the control objective (4).

Proof. Substituting (25) into (24), we have,

$$\dot e_1=-k_1e_1, \dot e_2=-k_2e_2, \dot e_3=-k_3e_3 \tag{27}$$

Therefore, the error $[e_1, e_2, e_3]^T$ would converge to zero with an exponential decaying rate [8]. According to the Lemma 1, we conclude that the control objective (4) is achieved. □

Within the proposed control scheme above, the pose tracking/stabilization errors can be steered converging to a ball containing the origin, as shown in Figure 1. Though the asymptotical stability is not achieved, the ultimate bound of pose errors can be arbitrarily small by choosing ε small enough, which can be seen from (23).

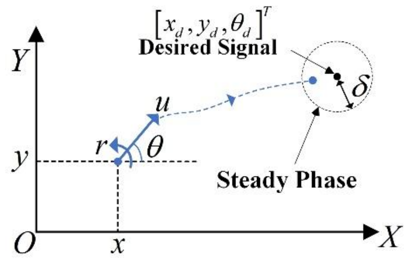


Figure 1 Pose error in steady state.

Remark 2. Note that the constant ε in (25) should be set according to the response speed of driven motors. For example, a faster motor response speed usually implies that we can choose a smaller ε and obtain better tracking accuracy.

3. Numerical simulation

In this section, numerical simulation is performed with three different types of desired pose signals to validate the control scheme. The control coefficients for all cases are identically selected as

$$k_1 = k_2 = k_3 = 5.5 \tag{28}$$

We choose the initial states of wheeled mobile robots by

$$x(0) = -6, y(0) = -3, \theta(0) = -0.5\pi \tag{29}$$

The desired pose signals are:

Case 1. Time-varying circular curve

$\dot{x}_d = 0.2\cos\theta_d, \dot{y}_d = 0.2\sin\theta_d, \dot{\theta}_d = 0.025$

$x_d(0) = 0, y_d(0) = -8, \theta_d(0) = 0$

Case 2. Time-varying straight line

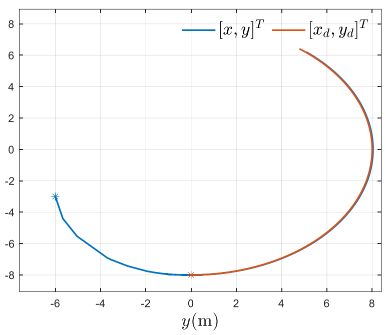
$\dot{x}_d = 0.2\cos\theta_d, \dot{y}_d = 0.2\sin\theta_d, \dot{\theta}_d = 0$

$x_d(0) = -3.5, y_d(0) = -3, \theta_d(0) = 0.25\pi$

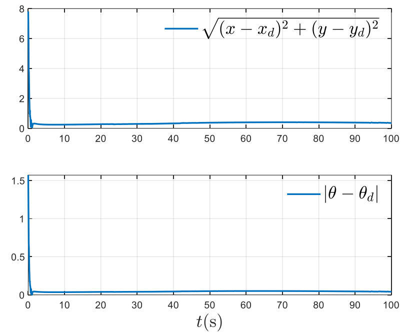
Case 3. Time-invariant pose

$x_d \equiv 0, y_d \equiv 0, \theta_d \equiv 0$

In particular, we set $\varepsilon = 0.05$ for the case 3 and depict the simulation results in Figures 2-4. As can be seen, the control laws (24)(25) can stabilize the NWMR into the neighborhood of reference pose signal that can be time-varying or time-invariant. Note also that the ultimate bound of pose control errors δ can be reduced by decreasing ε, which can be drawn from (23) in Lemma 1 and comparing the error trajectories in Figures 2-3 with that in Figure 4. Therefore, we conclude that all simulation results accord with the theoretical analysis.

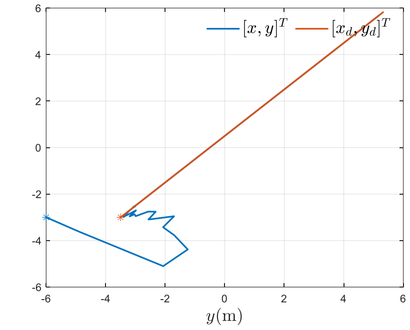


a. Position trajectories (*: starting point)

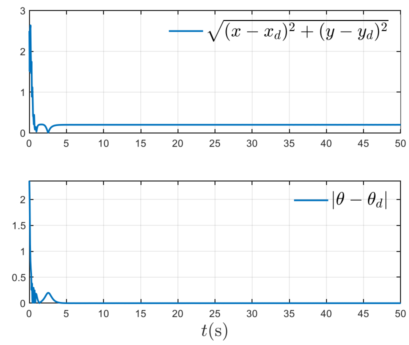


b. Pose error trajectories

Figure 2 Results of simulation case 1.

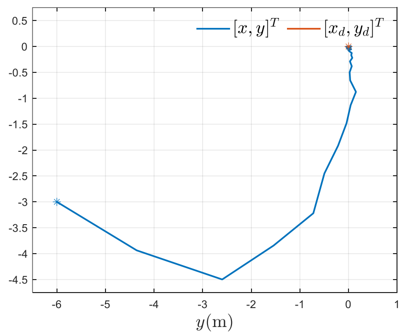


a. Position trajectories (*: starting point)

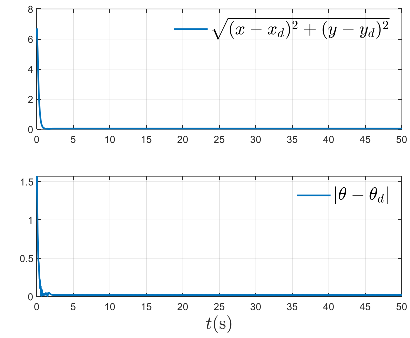


b. Pose error trajectories

Figure 3 Results of simulation case 2.



a. Position trajectories (*: starting point)



b. Pose error trajectories

Figure 4 Results of simulation case 3.

4. Conclusion

This work has investigated the practical linearization control for typical nonholonomic wheeled mobile robots. The underactuated property of the robot in question is compensated by a virtual control input after fusing the original pose states with an external dynamic oscillator. The feedback linearized form of the system is also illustrated. It is worth noting that the proposed control scheme enables stabilization and tracking via a single tracking control law. The pose tracking/stabilization errors are driven converging into a small ball centered by the origin. In the future, we will apply the current work to solve the formation tracking control of networked NWMRs.

5. Future recommendation

The tracking errors cannot be steered to arbitrarily small values when applying the proposed control law in practical wheeled mobile robots, because the real driven motors cannot follow the velocity commands that are too fast. In the future, we would like to investigate new techniques to attenuate the reliance on motor response frequency for the practical linearization control approach. Moreover, we also plan to find the condition associated with reference trajectory and state transformations, on which the pose errors can be driven to converge to zero with the presented control law.

Acknowledgements

This work was supported in part by the National Natural Science Foundation of China (Nos. 62133001, 62227810, 62403037) and the National Basic Research Program of China (973 Program: 2012CB821200, 2012CB821201).

References

1. L. Yan and B. Ma, Adaptive practical leader-following formation control of multiple nonholonomic wheeled mobile robots, International Journal of Robust Nonlinear Control, Vol.30(17), 2020, pp.7216-7237.
2. X. Zhang, Y. Fang, and N. Sun, Visual servoing of mobile robots for posture stabilization: from theory to experiments, International Journal of Robust and Nonlinear Control, Vol. 25(1), pp.1-15, 2015.
3. A. Astolfi, Discontinuous control of nonholonomic systems, Systems and Control Letters, Vol.27(1), pp.37-45, 1996.
4. R. Fierro and F. L. Lewis, Control of a nonholonomic mobile robot: Backstepping kinematics into dynamics, Journal of Robotic Systems, Vol.14(3), pp.149-163, 1998.
5. L. Yan and B. Ma, Universal Control for Both Rendezvous and Tracking of Multiple Nonholonomic Unicycles, IEEE Transactions on Control of Network Systems, Vol.11(1), 2024, pp. 439-449.
6. W. Xie and B. Ma, Position centroid rendezvous and centroid formation of multiple unicycle agents, IET Control Theory and Applications, Vol.8(17), pp.2055–2061, 2014.
7. M. A. Maghenem, A. Loria, and E. Panteley, Cascades-based leaderfollower formation tracking and stabilization of multiple nonholonomic vehicles, IEEE Transactions on Automatic Control, Vol.65(8), pp. 3639–3646, 2019.
8. H.K., Khalil, Nonlinear Systems (3rd Edition), Patience Hall, 2002.

Authors Introduction

Dr. Lixia Yan

He received B.S. degree from Beijing Jiaotong University, Beijing, China in 2013, and the M.S. and Ph.D degrees from Beihang University, Beijing, China in 2016 and 2021, respectively. He is currently an associate researcher affiliated with the Department of Automation at Tsinghua University. His research interests include nonlinear control theory, multiagent systems and embedded system applications. He is a member of IEEE.

Prof. Yingmin Jia

He received B.S. degree from Shandong University, Jinan, China, in 1982, and the M.S. and Ph.D. degrees from Beihang University, Beijing, China, in 1990 and 1993, respectively.

He is currently a professor affiliated with the Seventh Research Division and the director of the Center for Information and Control at Beihang University. His research interests include robust control, robotic systems, spacecraft coordination and on-orbit servicing. He is a senior member of IEEE.