

## Research Article

# Time-Varying Lyapunov Function for Mechanical Systems

 Bin Zhang<sup>1,\*</sup>, Yingmin Jia<sup>2</sup>
<sup>1</sup>School of Automation, Beijing University of Posts and Telecommunications, Haidian District, Beijing 1008760, China

<sup>2</sup>The Seventh Research Division and the Center for Information and Control, School of Automation Science and Electrical Engineering, Beijing University (BUAA), Haidian District, Beijing 100191, China

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## ABSTRACT

In this paper, a general method for constructing time-varying Lyapunov functions is provided for mechanical systems. A new class of time-varying Lyapunov functions is provided and sufficient conditions of uniform asymptotical stability are established. Different from the existing results on this subject, we remove the periodic restrictions and persistency of excitation restrictions in our work. Moreover, the results are extended into the robustness cases with unmodeled dynamics. Based on the Lyapunov methods, it is shown that uniform asymptotical stability can be obtained by using the control input provided in our work.

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## 1. INTRODUCTION

In this paper, we investigate the stability problem of time-varying mechanical systems. It is well known that Lyapunov functions provide fundamental tools for stability analysis and controller design. In terms of Lyapunov functions, two research directions are widely investigated, that is, converse Lyapunov theory and constructions of Lyapunov functions. Converse Lyapunov theory guarantees the existence of smooth Lyapunov functions for some systems. However, construction methods give specific structures of Lyapunov function candidates. In this paper, we deal with the second research topic: construction methods. We aim to provide a generic approach for finding strict Lyapunov functions of nonlinear time-varying mechanical systems described by Euler–Lagrange equation. Several approaches for construction of strict Lyapunov functions have been proposed under various frameworks and control objectives [1,2].

In Han et al. [3], a new method for estimation the domain of attraction was shown by using non-strict Lyapunov functions. In Bretas and Alberto [4], an extension of invariance principle with non-strict Lyapunov functions was provided for power systems with transmission losses. In Hong et al. [5], strict Lyapunov functions were established for finite-time control problem of robot manipulators. In Zhang and Jia [6], two kinds of weak-invariance principles for nonlinear switched systems were developed and accurate convergent regions were obtained. The readers can refer to Malisoff and Mazenc [7], Mazenc [8], and Giesl and Hafstein [9] for more detailed discussions on this topic and a list of related references.

As we all know, some approaches have been well established to build explicit strict Lyapunov functions for time-invariant systems, such as variable gradient method, Krasovskii method and so on. But construction of strict Lyapunov functions for time-varying systems is a challenging problem and there are no general methods.

Motivated by the above considerations, we provide a general method for constructing time-varying Lyapunov functions of mechanical systems. We first consider a mechanical system with time-varying frictions. With the control input, the closed-loop system is obtained. Then, we divide the closed-loop system into two subsystems. For subsystem one, it is shown that the origin is uniformly asymptotically stable and a class of time-varying Lyapunov function  $V(t, x)$  is established. For the subsystem two, it is shown that the derivative of  $V(t, x)$  is bounded. Based on the derivatives of the time-varying Lyapunov function along the two different subsystems, we provide the time-varying Lyapunov function for the whole system. It is shown uniform asymptotical stability can be obtained with the control input provided in this paper.

## 2. PROBLEM STATEMENT

Consider the following mechanical system with time-varying friction [Equation (1)]

$$M\ddot{q} + C\dot{q} + (F_c(t) + F_s(t)e^{-\rho\mu(\dot{q})})\text{sat}(\dot{q}) + k(t)q = \tau \quad (1)$$

where  $q(t), \dot{q}(t), \ddot{q}(t) \in R^n$  represent the position, velocity, and acceleration, respectively;  $M \in R^{n \times n}$  is the positive definite inertia of the system;  $\rho$  is a positive constant;  $\mu(\cdot)$  is a positive definite function related to the Stribeck effect;  $k(t)$  denotes a time-varying

\*Corresponding author. Email. [zb362301@126.com](mailto:zb362301@126.com)

spring stiffness. For system (1), it is assumed that the following properties hold.

**Property 1.** The matrix  $M$  is positive definite. Moreover, we define  $\lambda_{\max}$  as the maximum eigenvalue of  $M$  and  $\lambda_{\min}$  as the minimum eigenvalue of  $M$ .

**Property 2.** The derivative of  $k(t)$  is bounded such that [Equation (2)]

$$|k(t)| \leq \hat{k} \quad \forall t \geq 0 \tag{2}$$

where  $\hat{k}$  is a positive constant.

Before presenting the main results, we give some lemmas first.

**Lemma 1.** Zhang et al. [10] Let  $a, b, p, q$  be positive real numbers. If  $\frac{1}{p} + \frac{1}{q} = 1$ , then [Equation (3)]

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{3}$$

**Lemma 2.** Zhang et al. [10] Let  $a_i > 0, b_i > 0, i = 1, 2, \dots, n, p > 1$ , and  $q > 1$ . If  $\frac{1}{p} + \frac{1}{q} = 1$ , then [Equation (4)]

$$\sum_{i=1}^n a_i b_i \leq \left( \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n b_i^q \right)^{\frac{1}{q}} \tag{4}$$

**Lemma 3.** Zhang et al. [10] For  $x_i \in R, i = 1, 2, \dots, n$ , and  $0 < m < 1$ , then the following inequality holds [Equation (5)]:

$$|x_1|^m + \dots + |x_n|^m \leq (|x_1| + \dots + |x_n|)^m \tag{5}$$

**Lemma 4.** For  $x_i \in R, i = 1, 2, \dots, n$ , and  $0 < m < 1$ , then the following inequality holds [Equation (6)]:

$$|x_1|^m + \dots + |x_n|^m \leq n^{1-m} (|x_1| + \dots + |x_n|)^m \tag{6}$$

**Proof.** Let  $a_i = |x_i|^m, b_i = 1, i = 1, 2, \dots, n, p = \frac{1}{m}$ , and  $q = \frac{1}{1-m}$ .

Then based on Lemma 2, we obtain Equation (7)

$$\begin{aligned} & |x_1|^m + \dots + |x_n|^m \\ & \leq \left( \sum_{i=1}^n (|x_i|^m)^{\frac{1}{m}} \right)^m \left( \sum_{i=1}^n 1^{1-m} \right)^{1-m} \\ & = n^{1-m} (|x_1| + \dots + |x_n|)^m \end{aligned} \tag{7}$$

which completes the proof.

### 3. SUBSYSTEMS

For the uniform asymptotic stability problem of time-varying mechanical system (1), we propose the following control input [Equation (8)]

$$\tau = -\alpha M \dot{q} + C \dot{q} - k_0 q \tag{8}$$

where  $\alpha$  and  $k_0$  are control parameters. Let  $q = x_1$  and  $\dot{q} = x_2$ . Then the closed-loop system (1) and (8) can be written in a compact form as Equation (9)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(k(t) + k_0)M^{-1}x_1 - \alpha x_2 \\ &\quad - M^{-1}(F_c(t) + F_s(t)e^{-\rho\mu(x_2)})\text{sat}(x_2) \end{aligned} \tag{9}$$

We define [Equation (10)]

$$f(t, x) = \begin{pmatrix} x_2 \\ \tilde{f}(t, x) \end{pmatrix} \tag{10}$$

where function  $\tilde{f}(t, x) = -(k(t) + k_0)M^{-1}x_1 - \alpha x_2 - M^{-1}(F_c(t) + F_s(t)e^{-\rho\mu(x_2)})\text{sat}(x_2)$ . We also define  $f_1(t, x) = \begin{pmatrix} x_2 \\ -(k(t) + k_0)M^{-1}x_1 - \alpha x_2 \end{pmatrix}$

and  $f_2(t, x) = \begin{pmatrix} 0 \\ -M^{-1}(F_c(t) + F_s(t)e^{-\rho\mu(x_2)})\text{sat}(x_2) \end{pmatrix}$ . Then, we can see that  $f(t, x) = f_1(t, x) + f_2(t, x)$ . In this subsection, we consider two subsystems  $\dot{x} = f_1(t, x)$  and  $\dot{x} = f_2(t, x)$ , respectively.

Inspired by the results in Malisoff and Mazenc [7], we consider the following Lyapunov function [Equation (11)]

$$V(t, x) = A((k(t) + k_0)x_1^T x_1 + x_2^T M x_2) + x_1^T x_2 \tag{11}$$

where  $A > 0$  is a positive constant. In this paper, we choose

$$A > \max \left\{ \frac{1}{\lambda_{\min}}, \frac{1}{k_1 + k_0} \right\}.$$

It can be verified directly that [Equation (12)]

$$\begin{aligned} V(t, x) &= A((k(t) + k_0)x_1^T x_1 + x_2^T M x_2) + x_1^T x_2 \\ &> -\frac{1}{2}(x_1^T x_1 + x_2^T x_2) \end{aligned} \tag{12}$$

The derivative of  $V(t, x)$  along subsystem  $\dot{x} = f_1(t, x)$  shows [Equation (13)]:

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f_1(t, x) \leq -l_1 x_1^T x_1 - l_2 x_2^T x_2 \tag{13}$$

where we have  $l_1 = (k_1 + k_0)\frac{1}{\lambda_{\max}} - \frac{1}{2}\alpha - A\hat{k}$  and  $l_2 = \frac{3}{2}\alpha - 1$ .

Therefore, we can see that if we choose  $\alpha > \frac{2}{3}$  and  $k_0 > \left(\frac{1}{2}\alpha + A\hat{k}\right)$

$\lambda_{\max} - k_1$ , subsystem  $\dot{x} = f_1(t, x)$  is uniformly asymptotically stable.

Similarly, the derivative of  $V(t, x)$  along subsystem  $\dot{x} = f_2(t, x)$  shows [Equation (14)]:

$$\frac{\partial V}{\partial x} f_2(t, x) \leq 2\sqrt{n} \frac{1}{\lambda_{\min}} F_c(t) + F_s(t) V^{\frac{1}{2}} \tag{14}$$

In the following, we define time-varying parameter  $w(t) = \sqrt{n} \frac{1}{\lambda_{\min}} \|F_c(t) + F_s(t)\|$ .

### 4. MAIN RESULTS

From (13) and (14), we can see that  $V(t, x)$  is not a Lyapunov function for system (1), because it is not easy to verify the negativity of

the derivative of  $V(t, x)$  along system  $\dot{x} = f(t, x)$ , which is shown as following Equation (15)

$$\dot{V}(t, x)|_{f(t, x)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f_1(t, x) + f_2(t, x)) \quad (15)$$

Therefore, motivated by the results in Malisoff and Mazenc [7], we provide the following time-varying Lyapunov function [Equation (16)]

$$V^\#(t, x) = \int_0^V \frac{1}{(1+s)^{0.5}} ds + \phi \int_{t_0}^t \left( \frac{1}{T} \int_\tau^{\tau+T} w(v) dv - w(\tau) \right) d\tau \ln(1+V) \quad (16)$$

where  $T, \phi$  are positive constants. It can be verified directly that if  $k_0$  and  $\alpha$  are chosen large enough, then there exist  $T$  and  $\phi$  such that [Equation (17)]

$$\dot{V}^\#(t, x)|_{f(t, x)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f_1(t, x) + f_2(t, x)) < 0 \quad (17)$$

Therefore, we conclude with the following theorem.

**Theorem 1.** Consider time-varying system (1). If the control input is chosen as (8) with control parameters  $\alpha$  and  $k_0$  large enough, then the closed loop system is uniformly asymptotically stable. Moreover, (16) is a strict Lyapunov function.

**Proof.** This result can be verified by taking the derivative of  $V^\#(t, x)$  given in (16) directly. The detailed process is omitted here.

Then, we can extend the result to the robustness case. We take the following systems with unmodeled dynamics [Equation (18)]

$$M\ddot{q} + C\dot{q} + (F_c(t) + F_s(t)e^{-\rho\mu(\dot{q})})\text{sat}(\dot{q}) + k(t)q + H(q, \dot{q}) = \tau \quad (18)$$

where  $H(q, \dot{q})$  is the unmodeled dynamics. For system (18), we assume that [Equation (19)]

$$\lim_{\lambda \rightarrow 0} \frac{H(t, \lambda q, \lambda \dot{q})}{\lambda} = 0 \quad (19)$$

Without detailed proof, we provide the following theorem for the robustness case.

**Theorem 2.** Consider system (18) with unmodeled dynamics. We assume that (19) is satisfied. If the conditions in Theorem 1 hold, then we have that the closed-loop system is uniformly asymptotically stable.

## 5. NUMERICAL EXAMPLE

We give a numerical example to demonstrate the effectiveness of the proposed theoretical results under different initial values. Two different initial values are chosen as  $x(0) = (0.5, 0.3)$  and  $x(0) = (-0.1, 0.05)$ , respectively. By using the control input (8), we can see that the state trajectories converge to the origin asymptotically (Fig. 1). The simulation clearly illustrate the exactness of our results.

## 6. CONCLUSION

The stability problems have been considered in this paper for time-varying mechanical systems. A class of strict Lyapunov

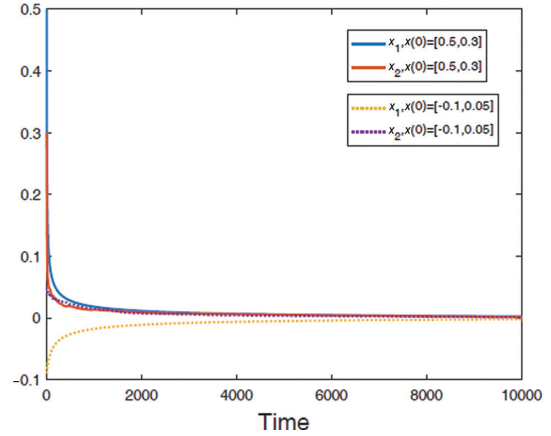


Figure 1 | The state trajectories of system.

functions has been established, in which time-varying terms have been included. By using the new Lyapunov functions, concise criterion for uniform asymptotical stability have been provided. A numerical example has been presented for illustration.

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## Authors Introduction

**Dr. Bin Zhang**



He received his PhD degrees in control theory and applications from Beihang University, in 2016. He is with the School of Automation, Beijing University of Posts and Telecommunication. His research interests include multi-agent systems.

**Dr. Yingmin Jia**



He received the B.S. degree in control theory from Shandong University, Jinan, China, in 1982, and the M.S. and PhD degrees in control theory and applications from Beihang University, Beijing, China, in 1990 and 1993, respectively. In 1993, he joined the Seventh Research Division, Beihang University, where he is currently a Professor of automatic control.