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Research Article

Synchronization of Novel 5D Hyperchaotic Systems Based on Center Translation Method

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ABSTRACT

In this paper, synchronization of novel five-dimensional (5D) autonomous hyperchaotic systems is studied. The synchronization control law is proposed based on the center translation method. A structure compensator is formulated to make the mathematical model of the error system the same as that of the response system, and a linear feedback controller and its simplification are designed via the Lyapunov stability theory to make the error system globally asymptotically stable at the origin. Thus, the two 5D hyperchaotic systems with different initial values are synchronized. Some relevant numerical simulation results, such as the curves of the corresponding synchronization state variables and the errors, are given to illustrate the feasibility and effectiveness of the synchronization control law.

1. Introduction

Hyperchaos was first presented in 1979 by Otto Rössler [1]. The main differences between hyperchaotic system and chaotic system are as follows. Firstly, the minimal dimension of the phase space that embeds a hyperchaotic attractor should be at least four, which requires the minimum number of coupled first-order autonomous ordinary differential equations to be four. Secondly, the number of terms in the coupled equations giving rise to instability should be at least two, of which at least one should have a nonlinear function [2]. Hence, hyperchaos is much more complicated than chaos, and hyperchaos synchronization has greater application significance and engineering value in secure communication.

Stability control of the novel 5D hyperchaotic system has been presented in [2]. In this paper, the mathematical model of the novel 5D hyperchaotic system is given as the drive system. Hyperchaos synchronization of the 5D systems with different initial values is studied based on the center translation method, so that the mathematical model of the synchronization error system would be the same as that of the controlled system formulated in [2]. Thus, the control law in [2] can be applied to the design of the linear feedback synchronization controller in this paper. Corresponding numerical simulation results are presented to demonstrate the validity of the synchronization method.

2. The Novel 5D Hyperchaotic System

The dynamic equations of the novel 5D hyperchaotic system are

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$$\begin{aligned}\dot{x}_1 &= a(y_1 - x_1),\\ \dot{y}_1 &= (c-a)x_1 + cy_1 + w_1 - x_1 z_1,\\ \dot{z}_1 &= -bz_1 + x_1 y_1,\\ \dot{v}_1 &= mw_1,\\ \dot{w}_1 &= -y_1 - hv_1,\end{aligned} \tag{1}$$

where $x_1, y_1, z_1, v_1, w_1 \in R$ are state variables, and a = 23, b = 3, c = 18, m = 12 and h = 4 [2].

Let the initial values of the system (1) be (x_{10}, y_{10}, z_{10}, v_{10}, w_{10}) = (1, 1, 1, 1, 1), then the Lyapunov exponents respectively are $\lambda_{11} = 0.8732 > 0$, $\lambda_{12} = 0.1282 > 0$, $\lambda_{13} = -0.0013 \approx 0$, $\lambda_{14} = -0.5770 < 0$ and $\lambda_{15} = -8.4231 < 0$. It indicates that the system (1) is hyperchaotic. The attractors of the 5D hyperchaotic system (1) are shown in Figure 1.

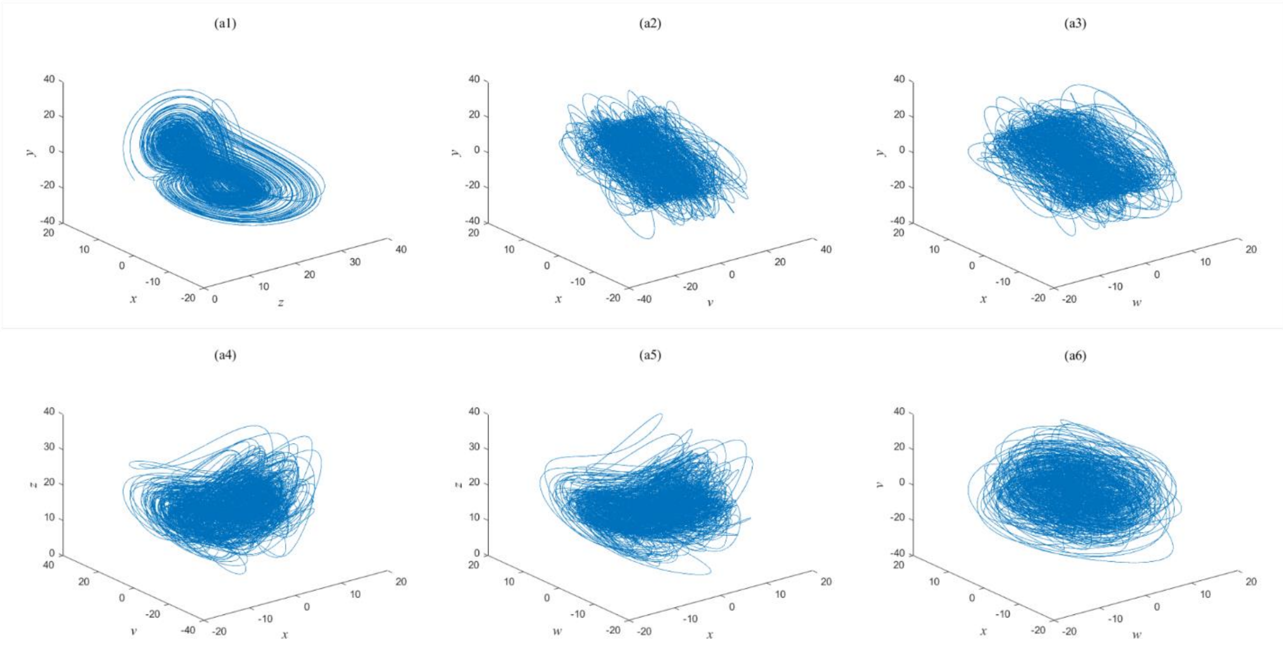


Figure 1 Attractors of the 5D hyperchaotic system: (a1) *z-x-y*; (a2) *v-x-y*; (a3) *w-x-y*; (a4) *x-v-z*; (a5) *x-w-z*; (a6) *w-x-v*

3. Hyperchaos Synchronization Based on Center Translation Method

3.1. *Formulation of error system*

Take the system (1) as the drive system, then the response system is formulated as

$$\begin{aligned}\dot{x}_2 &= a(y_2 - x_2) + u_{s1} + u_{c1},\\ \dot{y}_2 &= (c-a)x_2 + cy_2 + w_2 - x_2 z_2 + u_{s2} + u_{c2},\\ \dot{z}_2 &= -bz_2 + x_2 y_2 + u_{s3} + u_{c3},\\ \dot{v}_2 &= mw_2 + u_{s4} + u_{c4},\\ \dot{w}_2 &= -y_2 - hv_2 + u_{s5} + u_{c5},\end{aligned} \tag{2}$$

where

$$\boldsymbol{u}_s = \begin{bmatrix} u_{s1} & u_{s2} & u_{s3} & u_{s4} & u_{s5}\end{bmatrix}^{\mathrm{T}}$$

and

$$\boldsymbol{u}_c = \begin{bmatrix} u_{c1} & u_{c2} & u_{c3} & u_{c4} & u_{c5}\end{bmatrix}^{\mathrm{T}}$$

are structure compensator and synchronization controller to be designed. Let $\boldsymbol{u}_s = \mathbf{0}$, $\boldsymbol{u}_c = \mathbf{0}$, and the initial values of the response system (2) be (x_{20}, y_{20}, z_{20}, v_{20}, w_{20}) = (5, 0, 4, 3, 8), then the Lyapunov exponents respectively are $\lambda_{21} = 0.9121 > 0$, $\lambda_{22} = 0.1175 > 0$, $\lambda_{23} = -0.0008 \approx 0$, $\lambda_{24} = -0.5533 < 0$ and $\lambda_{25} = -8.4755 < 0$. It shows that the response system (2) is also hyperchaotic.

Let

$$\begin{aligned}\boldsymbol{e} &= \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5\end{bmatrix}^{\mathrm{T}}\\ &= \begin{bmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 & v_2 - v_1 & w_2 - w_1\end{bmatrix}^{\mathrm{T}}\end{aligned}$$

be the synchronization error and

$$\begin{aligned}\boldsymbol{u}_s &= \begin{bmatrix} u_{s1} & u_{s2} & u_{s3} & u_{s4} & u_{s5}\end{bmatrix}^{\mathrm{T}} \\ &= \begin{bmatrix} 0 \\ x_2 z_1 + x_1 z_2 - 2x_1 z_1 \\ -x_2 y_1 - x_1 y_2 + 2x_1 y_1 \\ 0 \\ 0 \end{bmatrix},\end{aligned}$$

then the error system is simplified as

$$\begin{aligned}\dot{e}_1 &= a\left(e_2 - e_1\right) + u_{c1}, \\ \dot{e}_2 &= \left(c-a\right)e_1 + ce_2 + e_5 - e_1 e_3 + u_{c2}, \\ \dot{e}_3 &= -be_3 + e_1 e_2 + u_{c3}, \\ \dot{e}_4 &= me_5 + u_{c4}, \\ \dot{e}_5 &= -e_2 - he_4 + u_{c5}. \end{aligned} \tag{3}$$

Comparing the mathematical model of the error system (3) with that of the controlled system (2) in [2], it can be found that the two models are similar. Hence, the synchronization controller $\boldsymbol{u}_c$ is designed as

$$\begin{aligned}\boldsymbol{u}_c &= \begin{bmatrix} u_{c1} & u_{c2} & u_{c3} & u_{c4} & u_{c5}\end{bmatrix}^{\mathrm{T}} \\ &= \begin{bmatrix} -k_1 e_1 & -k_2 e_2 & -k_3 e_3 & -k_4 e_4 & -k_5 e_5\end{bmatrix}^{\mathrm{T}},\end{aligned}$$

where $k_1, k_2, k_3, k_4, k_5 \geq 0$.

3.2. *Design of linear feedback synchronization controller*

Theorem 1. *Let* $\boldsymbol{x} = \boldsymbol{0}$ *be an equilibrium point for* $\dot{\boldsymbol{x}} = \boldsymbol{f}\left(\boldsymbol{x}\right)$, *where* $\boldsymbol{f}: D \to R^n$ *is a locally Lipschitz map from a domain* $D \subset R^n$ *into* R^n. *Let* $V: R^n \to R$ *be a continuously differentiable function such that*

$$V\left(\boldsymbol{0}\right) = 0 \;\text{ and }\; V\left(\boldsymbol{x}\right) > 0, \;\; \forall \boldsymbol{x} \neq \boldsymbol{0}$$

$$\left\|\boldsymbol{x}\right\| \to \infty \Rightarrow V\left(\boldsymbol{x}\right) \to \infty$$

$$\dot{V}\left(\boldsymbol{x}\right) < 0, \;\; \forall \boldsymbol{x} \neq \boldsymbol{0}$$

then $\boldsymbol{x} = \boldsymbol{0}$ *is globally asymptotically stable* [2].

Take a continuously differentiable function

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + \frac{h}{m}e_4^2 + e_5^2\right) \tag{4}$$

as a Lyapunov function candidate for the error system (3). Then, the derivative $\dot{V}$ is derived as

$$\begin{aligned}\dot{V} &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + \frac{h}{m}e_4\dot{e}_4 + e_5\dot{e}_5 \\ &= -\left(k_1 + a\right)e_1^2 + ce_1 e_2 - \left(k_2 - c\right)e_2^2 \\ &\quad -\left(k_3 + b\right)e_3^2 - k_4\frac{h}{m}e_4^2 - k_5 e_5^2 \\ &\leq -\left(k_1 + a - \frac{c}{2}\right)e_1^2 - \left(k_2 - \frac{3}{2}c\right)e_2^2 \\ &\quad -\left(k_3 + b\right)e_3^2 - k_4\frac{h}{m}e_4^2 - k_5 e_5^2. \end{aligned} \tag{5}$$

For $\dot{V} < 0$, the parameters k_1, k_2, k_3, k_4 and k_5 should satisfy that

$$\begin{matrix} k_1 + a - \dfrac{c}{2} > 0, \\ k_2 - \dfrac{3}{2}c > 0, \\ k_3 + b > 0, \\ k_4\dfrac{h}{m} > 0, \\ k_5 > 0, \end{matrix} \;\Rightarrow\; \begin{matrix} k_1 > \dfrac{c}{2} - a, \\ k_2 > \dfrac{3}{2}c, \\ k_3 > -b, \\ k_4 > 0, \\ k_5 > 0, \end{matrix} \;\Rightarrow\; \begin{matrix} k_1 = 0, \\ k_2 = 30, \\ k_3 = 0, \\ k_4 = 1, \\ k_5 = 1. \end{matrix}$$

Thus, the linear feedback synchronization controller $\boldsymbol{u}_c$ is designed as

$$\begin{aligned}\boldsymbol{u}_c &= \begin{bmatrix} u_{c1} & u_{c2} & u_{c3} & u_{c4} & u_{c5}\end{bmatrix}^{\mathrm{T}} \\ &= \begin{bmatrix} 0 & -30e_2 & 0 & -e_4 & -e_5\end{bmatrix}^{\mathrm{T}}. \end{aligned} \tag{6}$$

From Theorem 1, the error system (3) is globally asymptotically stable at the origin. It indicates that the response system (2) is synchronized with the drive system (1).

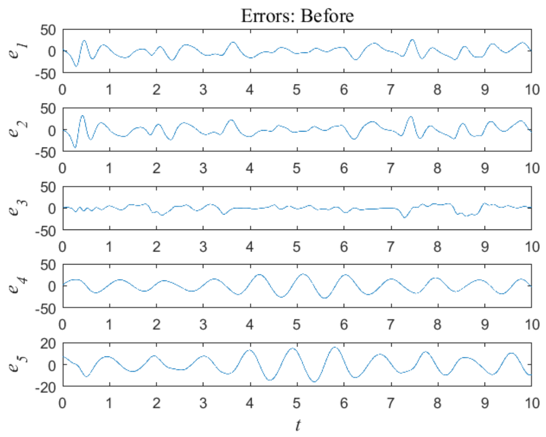
3.3. *Numerical simulation under the synchronization controller* $\boldsymbol{u}_c$

Remark 1. *The initial values of the drive system* (1) *and the response system* (2) *are* (x_{10}, y_{10}, z_{10}, v_{10}, w_{10}) = (1, 1, 1, 1, 1) *and* (x_{20}, y_{20}, z_{20}, v_{20}, w_{20}) = (5, 0, 4, 3, 8) *respectively in this paper.*

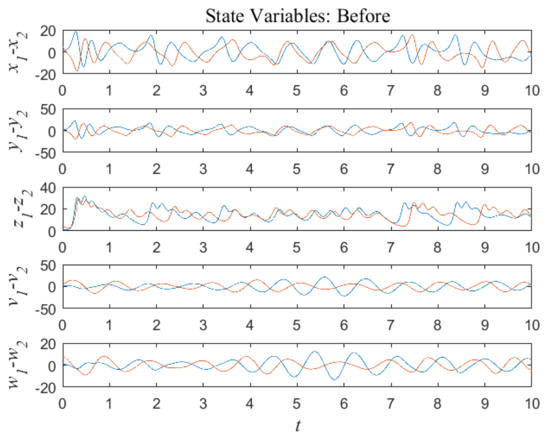
Definition 1. *After adding the structure compensator* $\boldsymbol{u}_s$ *and the linear feedback synchronization controller* $\boldsymbol{u}_c$ *to the response system* (2), *the Lyapunov exponents of the response system* (2) *are called sub-Lyapunov exponents* [3].

Theorem 2. *The response system* (2) *and the drive system* (1) *will synchronize only if the sub-Lyapunov exponents are all negative* [3].

The curves of the errors and the corresponding state variables before and after adding the structure compensator $\boldsymbol{u}_s$ and the linear feedback synchronization controller $\boldsymbol{u}_c$ to the response system (2) are shown in Figure 2 and Figure 3 respectively. Comparing Figure 3 with Figure 2, it can be found that the errors e_1, e_2, e_3, e_4 and e_5 converge to zero asymptotically and rapidly and the corresponding state variables are synchronized well after adding $\boldsymbol{u}_s$ and $\boldsymbol{u}_c$ to the response system (2). Moreover, the sub-Lyapunov exponents of the response system (2) are λ_{21c} = -1.0292, λ_{22c} = -1.0355, λ_{23c} = -3.0000, λ_{24c} = -17.4669 and λ_{25c} = -17.4690, which are all negative. From Theorem 2, the response system (2) and the drive system (1) have synchronized.

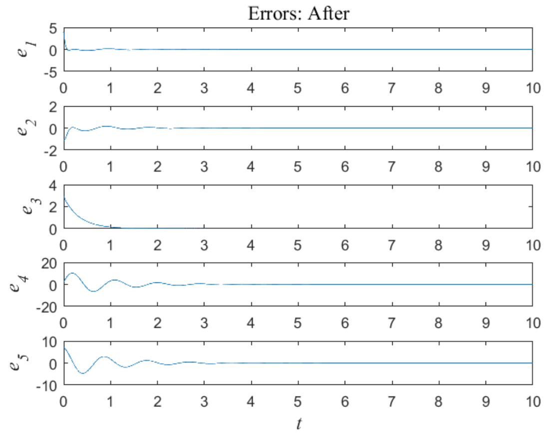


(a) Errors

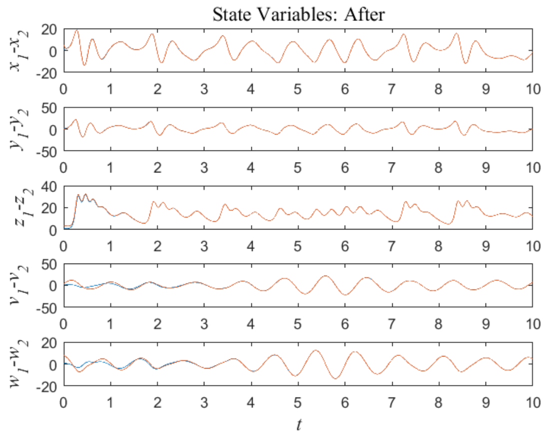


(b) State variables

Figure 2 Before: (a) Errors; (b) State variables



(a) Errors



(b) State variables

Figure 3 After adding $\boldsymbol{u}_s$ and $\boldsymbol{u}_c$: (a) Errors; (b) State variables

3.4. *Simplification of synchronization controller*

Corollary 1. *Let* $\boldsymbol{x} = \boldsymbol{0}$ *be an equilibrium point for* $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, *where* $\boldsymbol{f}: D \to R^n$ *is a locally Lipschitz map from a domain* $D \subset R^n$ *into* R^n. *Let* $V: R^n \to R$ *be a continuously differentiable, radially unbounded, positive definite function such that* $\dot{V}(\boldsymbol{x}) \leq 0$ *for all* $\boldsymbol{x} \in R^n$. *Let* $S = \left\{ \boldsymbol{x} \in R^n \middle| \dot{V}(\boldsymbol{x}) = 0 \right\}$ *and suppose that no solution can stay identically in S, other than the trivial solution* $\boldsymbol{x}(t) \equiv \boldsymbol{0}$. *Then, the origin is globally asymptotically stable* [2].

Still take Equation (4) as a Lyapunov function candidate for the error system (3). Now let $k_4 = 0$ and substitute $k_1 =$

$k_3 = k_4 = 0$ into Equation (5). Then, the derivative $\dot{V}$ is reduced to

$$\begin{aligned}\dot{V} &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + \frac{h}{m}e_4\dot{e}_4 + e_5\dot{e}_5 \\ &= -ae_1^2 + ce_1e_2 - (k_2 - c)e_2^2 - be_3^2 - k_5e_5^2 \\ &\le -\left(a - \frac{c}{2}\right)e_1^2 - \left(k_2 - \frac{3}{2}c\right)e_2^2 - be_3^2 - k_5e_5^2.\end{aligned} \tag{7}$$

For $\dot{V} \le 0$, the parameters k_2 and k_5 should satisfy that

$$\begin{matrix} k_2 - \frac{3}{2}c > 0, \\ k_5 > 0, \end{matrix} \Rightarrow \begin{matrix} k_2 > \frac{3}{2}c, \\ k_5 > 0, \end{matrix} \Rightarrow \begin{matrix} k_2 = 30, \\ k_5 = 1. \end{matrix}$$

From Corollary 1, to find $S = \left\{\boldsymbol{e} \in R^5 \middle| \dot{V}(\boldsymbol{e}) = 0\right\}$, note that

$$\dot{V} = 0 \Rightarrow e_1 = e_2 = e_3 = e_5 = 0$$

Hence, $S = \left\{\boldsymbol{e} \in R^5 \middle| e_1 = e_2 = e_3 = e_5 = 0\right\}$.

Let $\boldsymbol{e}(t)$ be a solution that belongs identically to $S = \left\{\boldsymbol{e} \in R^5 \middle| e_1 = e_2 = e_3 = e_5 = 0\right\}$, so that

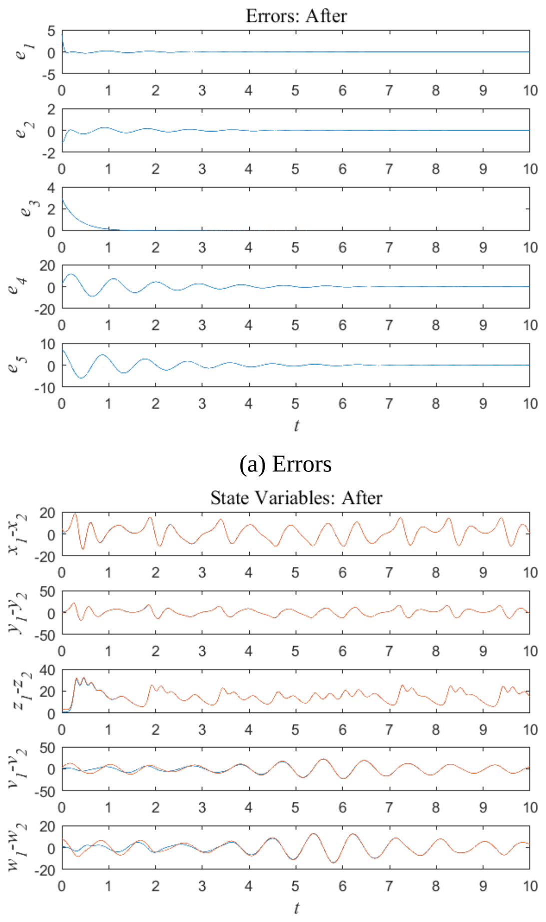
$$\begin{aligned} &e_1(t) = e_2(t) = e_3(t) = e_5(t) \equiv 0 \\ &\Rightarrow \dot{e}_1(t) = \dot{e}_2(t) = \dot{e}_3(t) = \dot{e}_4(t) = \dot{e}_5(t) \equiv 0 \\ &\Rightarrow e_4(t) \equiv 0 \end{aligned}$$

Therefore, the only solution that can stay identically in $S = \left\{\boldsymbol{e} \in R^5 \middle| \dot{V}(\boldsymbol{e}) = 0\right\}$ is the trivial solution $\boldsymbol{e}(t) \equiv \boldsymbol{0}$. From Corollary 1, the origin is globally asymptotically stable. Finally, the synchronization controller $\boldsymbol{u}_c$ in Equation (6) is simplified as

$$\begin{aligned}\boldsymbol{u}_{cs} &= \begin{bmatrix} u_{cs1} & u_{cs2} & u_{cs3} & u_{cs4} & u_{cs5} \end{bmatrix}^{\mathrm{T}} \\ &= \begin{bmatrix} 0 & -30e_2 & 0 & 0 & -e_5 \end{bmatrix}^{\mathrm{T}}. \end{aligned} \tag{8}$$

3.5. *Numerical simulation under the simplified synchronization controller $\boldsymbol{u}_{cs}$*

The curves of the errors and the corresponding state variables before and after adding the structure compensator $\boldsymbol{u}_s$ and the simplified linear feedback synchronization controller $\boldsymbol{u}_{cs}$ to the response system (2) are shown in Figure 2 and Figure 4 respectively. Comparing Figure 4 with Figure 2, it can be found that the errors e_1, e_2, e_3, e_4 and e_5 converge to zero asymptotically and rapidly and the corresponding state variables are synchronized well after adding $\boldsymbol{u}_s$ and $\boldsymbol{u}_{cs}$ to the response system (2). Furthermore, the sub-Lyapunov exponents of the response system (2) are λ_{21cs} = -0.5284, λ_{22cs} = -0.5302, λ_{23cs} = -3.0006, λ_{24cs} = -17.4668 and λ_{25cs} = -17.4690, which are all negative. From Theorem 2, the response system (2) and the drive system (1) have synchronized. However, comparing Figure 4 with Figure 3, it can be seen that the speed of convergence and synchronization under $\boldsymbol{u}_{cs}$ is a little bit slower than that under $\boldsymbol{u}_c$. Nevertheless, the simplified synchronization controller $\boldsymbol{u}_{cs}$ only has two feedback variables, such that it is easier to implement via circuit than the synchronization controller $\boldsymbol{u}_c$. Hence, the simplified synchronization controller $\boldsymbol{u}_{cs}$ has higher value in engineering application.



(a) Errors

(b) State variables

Figure 4 After adding $\boldsymbol{u}_s$ and $\boldsymbol{u}_{cs}$: (a) Errors; (b) State variables

4. Conclusions

Synchronization of the novel 5D hyperchaotic systems is proposed based on the center translation method in this paper. A linear feedback synchronization controller and its simplification are designed via the Lyapunov stability theory. Numerical simulation results illustrate the feasibility of the synchronization method. The study has some engineering significance. Furthermore, the circuit implementation of the synchronization system is under investigation and will be reported elsewhere.

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Author Introduction

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