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Research Article

Study of Synchronization of Two Heterogeneous Chaotic Systems

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ABSTRACT

This paper describes a method for synchronizing two chaotic systems with three-dimensional heterogeneous structures. We introduced a mathematical model and designed a new synchronous controller that can synchronize the system from different initial values. This paper analyzes the advantages and disadvantages of the synchronous controller, and also uses MATLAB software to generate the error curve of the synchronous system when the synchronous controller is activated on the response system, and shows its effectiveness.

1. Introduction

Research on chaos theory is an important subject in nonlinear phenomena.One of his early pioneers was the American scientist Lorentz, who demonstrated the existence of chaotic systems in 1963 [1]. Equations governing chaotic systems are significant, especially since there are chaotic solutions to nonlinear equations. We design a mathematical model suitable for a particular system, investigate whether chaotic behavior occurs under the model, and then verify the fundamental characteristics of chaotic systems.

The discovery of synchrony was first made by the famous physicist Huygens. Huygens happened to observe two adjacent pendulums swinging in perfect harmony. This breakthrough opened up disaster oscillator theory. The synchronous phenomenon in the natural world and the underlying mechanism have been brought to light. In 1990, Pecora and Carroll developed a response-driven synchronization method and demonstrated the phenomenon of chaotic synchronization for the first time in circuits by aligning the chaotic trajectories of the system under various initial conditions [2]. In physics, chaotic systems have been extensively studied and various methods have been developed. These include adaptive synchronization, delayed synchronization, pulse synchronization, etc. When investigating the synchronization of two chaotic systems, many researchers start by analyzing the equilibrium point of the control system. Then, both numerical simulations and functional methods are used to examine the accuracy of the simulation results based on the Lyapunov law stability theory [3], [4], [5]. We design an adaptive synchronous controller and propose an adaptive method to synchronize chaotic systems.

2. Mathematical models of chaotic systems

Chaos theory's Qi system can be represented mathematically as shown below:

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$$\begin{cases} \dot{x} = a\left(y - x\right) + yz \\ \dot{y} = cx - xz - y \\ \dot{z} = xy - bz \end{cases} \quad (1)$$

State variables of an unidentifiable nature, including variables like x, y, and z, are an integral part of the system. The state variable of the system is an unidentified quantity is $x, y, z \in \mathrm{R}$. $a = 35$, $b = 8/3$, $c = 80$ are the system's typical parameters.

The x, y, and z axes are the transformations for the state variables in the mathematical model $(x, y, z) \to (x, -y, -z)$, $(x, y, z) \to (-x, y, -z)$, $(x, y, z) \to (-x, -y, z)$.

The mathematical model for the system remains unchanged when subjected to z-axis transformation. Therefore, it may be concluded that the system's mathematical model is symmetric around the z-axis.

The mathematical model for the Qi chaotic system yields a partial derivative:

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -\left(a + b + 1\right) = -\frac{116}{3} < 0 \quad (2)$$

When a negative partial derivative is observed in the equation mentioned before, it indicates that the system's mathematical model is dissipative. And its trajectory is controlled within a finite boundary as time elapses.

2.1 Chaotic dynamic properties

Designate the starting value for the Qi chaotic system as $(x_0, y_0, z_0) = (1,1,1)$. The MATLAB function toolbox computes the Lyapunov index for the Qi chaotic system as $\lambda_1 = 4.0517 > 0$, $\lambda_2 = -0.0027 \approx 0$. The Lyapunov index holds significant importance since it $\lambda_1 = 4.0517 > 0$, this signifies that the Qi chaotic system can exhibit chaotic motion at that specific point in time. Calculating the Lyapunov dimension d_L of a system involves using the formula for solving by inputting the three Lyapunov exponents obtained:

$$d_L = j + \frac{\sum_{i=1}^{j} \lambda_i}{\left|\lambda_{j+1}\right|} = 2 + \frac{4.0517 - 0.0027}{42.7151} = 2.0948 \quad (3)$$

The given condition can be satisfied by finding the largest integer j that $\sum_{i=1}^{j} \lambda_i > 0$. Based on the formula mentioned above, it can be inferred that, d_L =2.0948 , thus, the dimensional value of the chaotic system is fractional in nature.

The graphs depicting the state variables of a chaotic system can be simulated using Matlab in the following manner. The Simulink simulation generates the phase trajectory diagram of the chaotic system as illustrated in Fig. 1. The phase trajectory curves for the Qi chaos system in varied coordinate systems are displayed in Fig. 2.

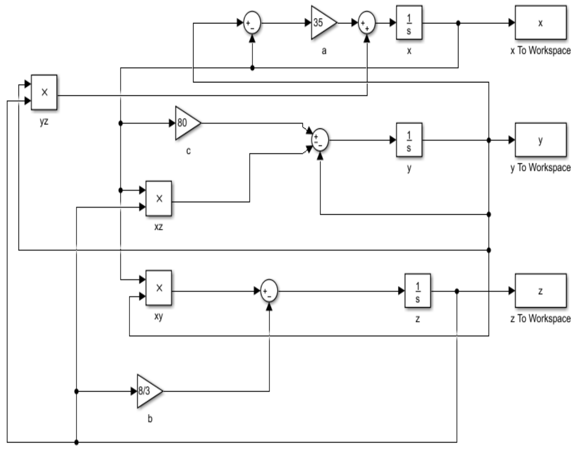
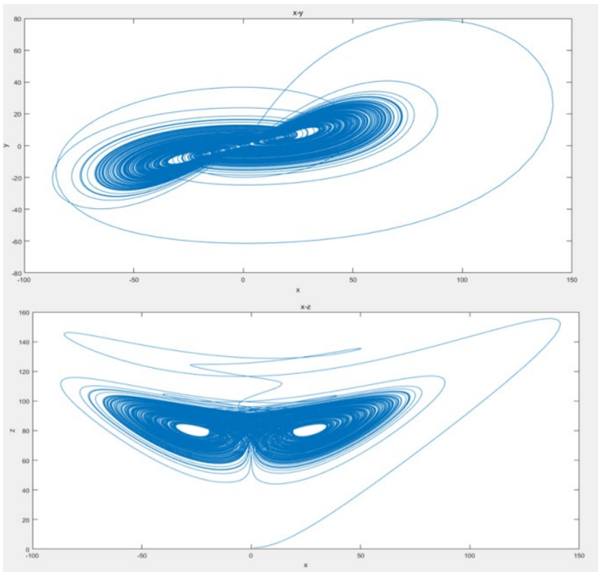


Fig. 1 The construction of the Qi chaotic system and its simulation using Simulink



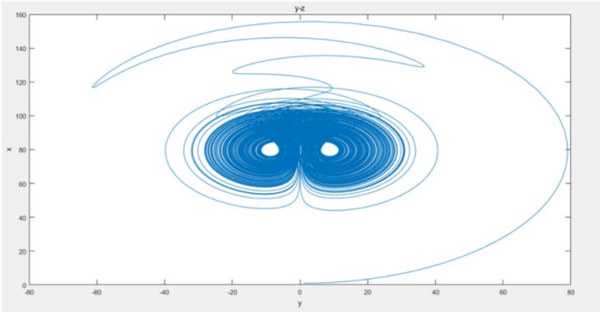


Fig. 2 Phase trajectory curve of Qi chaotic system system

Initiate the system by assigning initial values as follows $\left(x_{01}, y_0, z_0\right) = \left(1,1,1\right)$ and $\left(x_{02}, y_0, z_0\right) = \left(1.0001,1,1\right)$ respectively, and, a mathematical model can be constructed on Simulink to plot the solution curves of the system's three corresponding variables, as illustrated in Fig. 3.

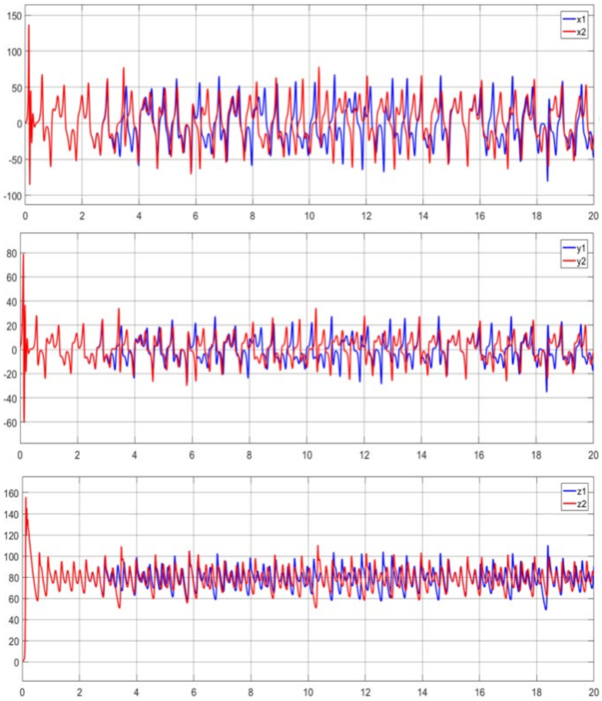


Fig. 3 Qi Solution curves of chaotic system in different states with different initial values

Under the same conditions, the solution curves of the Qi chaotic system will greatly differ in a short period of time upon modifying only the initial value of state variable x by 0.01%. It is evident that even a minimal change in initial value impacts the Qi chaotic system largely, indicating its susceptibility towards small perturbations. Sensitive dependence on initial conditions is a significant feature of chaotic systems, and its observation in the Qi chaotic system signifies its susceptibility towards even minute variations in initial value.

2.2 Characteristic of equilibrium point

The equilibrium state equation can be obtained by setting the right side of the system's mathematical model equation to 0:

$$\begin{cases} a\left(y-x\right)+yz=0 \\ cx-xz-y=0 \\ xy-bz=0 \end{cases} \tag{4}$$

Solve the system of state, let:

$$\begin{cases} x_0=\sqrt{\dfrac{b\left[ac+c^2-2a+c\sqrt{(a+c)^2-4a}\right]}{2a}} \\ y_0=\dfrac{\sqrt{2ab\left[ac+c^2-2a+c\sqrt{(a+c)^2-4a}\right]}}{a+c+\sqrt{(a+c)^2-4a}} \\ z_0=\dfrac{ac+c^2-2a+c\sqrt{(a+c)^2-4a}}{a+c+\sqrt{(a+c)^2-4a}} \end{cases} \tag{5}$$

To obtain the equilibrium point of the mathematical model of the system, the given parameters can be substituted into the formula as follows:

$$\begin{cases} S_1=\left(0,0,0\right) \\ S_2=\left(x_0,y_0,z_0\right) \\ S_3=\left(-x_0,-y_0,z_0\right) \end{cases} \rightarrow \begin{cases} S_1=\left(0,0,0\right) \\ S_2=\left(26.3899,8.0531,79.6948\right) \\ S_3=\left(-26.3899,-8.0531,79.6948\right) \end{cases} \tag{6}$$

Linearizing the Qi chaotic system at the equilibrium point involves writing the coefficient of each variable into the matrix to determine its Jacobian matrix, which helps evaluate the equilibrium point by substituting the values accordingly:

$$J_1=J\big|_{S_1}=\left.\begin{bmatrix} -a & a+z & y \\ c-z & -1 & -x \\ y & x & -b \end{bmatrix}\right|_{S_1}=\begin{bmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \tag{7}$$

Converting the aforementioned matrix into a determinant and equating it to zero helps obtain the characteristic equation upon its expansion, which is given as follows:

$$f\left(s\right)=\left(s+b\right)\left[s^2+\left(a+1\right)s-a\left(c-a\right)\right]=0 \tag{8}$$

By performing calculations, the characteristic roots of the matrix can be determined as follows:

$$s_1 = -b = -2.6667 \tag{9}$$

$$s_2 = \frac{-(a+1)+\sqrt{(a+1)^2+4a(c-1)}}{2} \approx 37.5788 \tag{10}$$

$$s_3 = \frac{-(a+1)-\sqrt{(a+1)^2+4a(c-1)}}{2} \approx -73.5788 \tag{11}$$

The equilibrium $S_1 = (0,0,0)$ is a saddle node.

The approach to analyzing the last two stable points is identical to that of S_1, and S_2 and S_3 exhibit symmetry along the z-axis. Therefore, by analyzing one of the equilibria, we can proceed to analyze S_2 equilibrium as well.

At equilibrium point $S_2 = (26.3899, 8.0531, 79.6948)$ (which is the same as S1), the J matrix of the Qi chaotic system is linearized

$$J_2 = \begin{bmatrix} -a & a+79.6948 & 8.0531 \\ c-79.6948 & -1 & -26.3899 \\ 8.0531 & 26.3899 & -b \end{bmatrix} \tag{12}$$

Using the aforementioned method, the matrix is converted to a determinant and then equated to zero. Next, the obtained determinant is expanded to obtain the characteristic equation, which yields the eigenvalues of J_2 through calculation:

$$s_1 = -45.8958 \tag{13}$$

$$s_{2,3} = 3.6146 \pm 32.3465\mathrm{j} \tag{14}$$

While S1 has a negative eigenvalue, the other two equilibrium points contain a pair of complex roots that are conjugate to each other with a positive real part. Applying the Routh stability criterion indicates that S_2 and S_3 have similar characteristics as unstable foci within the system.

From the data obtained in the table, we can conclude that only one of the three stable points shown by the Qi chaotic system remains stable, whereas the remaining two have been determined to be unstable. This instability leads to a divergence of nearby orbits, which becomes more pronounced over time. This divergence is a clear indication of the butterfly effect present within the system.

With its characteristic feature of dissipation, the Qi chaotic system is capable of producing chaotic motion, ensuring stability of the system as a whole. The presence of the dissipative property in the Qi chaotic system aids in stabilizing the system, leading to the convergence of outer orbits on the attractor. However, the nearby orbitals experience a repulsive force and must be exponentially separated. As a result individual parts of the system may be unstable, resulting in a complex and difficult structure.

The numerical simulation results demonstrate that the Lü chaotic system displays chaotic dynamics equivalent to that of the Qi chaotic system.

3. Design of synchronous controller

3.1 Driving system

This article adopts the Qi chaotic system as the driver and presents the mathematical expression for the Qi chaotic system model:

$$\begin{cases} \dot{x}_1 = a_1(y_1 - x_1) + y_1 z_1 \\ \dot{y}_1 = c_1 x_1 - x_1 z_1 - y_1 \\ \dot{z}_1 = x_1 y_1 - b_1 z_1 \end{cases} \tag{15}$$

Among them, the typical parameters of Qi chaotic system are: $a = 35$, $b = 8/3$, $c = 80$.

3.2 Response system

This article utilizes the Lü chaotic system as the responsive system, and the synchronization controller equation is substituted with equivalent words to create a synonymous sentence:

$$\begin{cases} \dot{x}_2 = a_2(y_2 - x_2) + u_{c1} \\ \dot{y}_2 = -x_2 z_2 + c_2 y_2 + u_{c2} \\ \dot{z}_2 = x_2 y_2 - b_2 z_2 + u_{c3} \end{cases} \tag{16}$$

Typical parameters of Lü chaotic system are: $a = 36$, $b = 3$, $c = 20$. The synchronization controller is $u_c = [u_{c1} \quad u_{c2} \quad u_{c3}]^T$.

To simulate the error system and the controller's state synchronization curve, MATLAB software was employed, determine if these two chaotic systems with different architectures can exhibit complete synchronization.

3.3 Direct method

When the system's equilibrium state is highly stable and its output has attained a state of balance. As time progresses, the energy contained within the system will gradually diminish until it reaches the stable and minimum value of the equilibrium state. The Lyapunov direct method is based on an energy perspective. The system's motion will result in energy consumption, but the system capacity will not be depleted to zero.

Lyapunov's method involves the use of an artificial function to achieve this goal, this function is called Lyapunov function, denoted as $v(x,t)$ or $v(x)$. Let $v(x)$ be any scalar function, where x is the unknown variable of the system, if $v(x)$ satisfies the following properties:

(1) $\dot{v}(x)=\frac{d_{v(x)}}{dt}$ is continuous and can reflect the trend of energy change;

(2) $v(x)$ is positive definite and can reflect the magnitude of energy;

(3) When $\|x\| \to \infty$, $v(x) \to \infty$ reflects the distribution of energy, function $v(x)$ is called Lyapunov function.

Given two systems: $A=(x_1, y_1, z_1)$, $Y=(x_2, y_2, z_2)$, u_c indicates the control quantity.

$$\begin{cases} A = f(A) \\ B = g(B) + u_c \end{cases} \tag{17}$$

The key to achieving synchronization between two disparate chaotic systems lies in the discovery of a suitable u_c that make $\lim_{t\to\infty} \|y(t)-x(t)\| = 0$.

For any initial values $x(0)$ and $y(0)$, the problem of controlling the synchronization of the system can be converted into a problem of system error through transformation.

Set the error $e=[e_1\ e_2\ e_3]^T=[x_2-x_1\ y_2-y_1\ z_2-z_1]^T$, choose the Lyapunov function $v(x)=\frac{1}{2}\sum_{i=1}^{n} e_i^2$, obviously $v(x)$ is positive definite, the state error equation will be rendered asymptotically stable at the origin if $v(x)$ is negative definite, that is, $\dot{v}(x)=\dot{e}_1e_1+\dot{e}_2e_2+\dot{e}_3e_3=0$ at $t\to\infty$ identification of a suitable uc is imperative to attain negative definiteness in $v(x)$.

Set the controller to:

$$u_C = [u_{c1}\quad u_{c2}\quad u_{c3}]^T \tag{18}$$

The error system can be formulated through substituting similar words in the mathematical models of the drive system and the response system:

$$\begin{aligned} \dot{e}_1 &= a_2(e_2-e_1)+(a_2-a_1)(y_1-x_1)-y_1z_1+u_{c1} \\ \dot{e}_2 &= -x_2z_2+c_2e_2+(c_2+1)y_1+x_1z_1-c_1x_1+u_{c2} \\ \dot{e}_3 &= x_2y_2-b_2e_3-x_1y_1+(b_1-b_2)z_1+u_{c3} \end{aligned} \tag{19}$$

Choose the Lyapunov function $v(x)=\frac{1}{2}\sum_{i=1}^{n} e_i^2$ and calculate $\dot{v}(x)=\dot{e}_1e_1+\dot{e}_2e_2+\dot{e}_3e_3$.

$$u_c = \begin{bmatrix} (a_2-1)e_1-a_2e_2+(a_2-a_1)(x_1-y_1)+y_1z_1 \\ x_2z_2-(1+c_2)e_2-(c_2+1)y_1-x_1z_1+c_1x_1 \\ x_1y_1+(b_2-1)e_3-x_2y_2+(b_2-b_1)z_1 \end{bmatrix} \tag{20}$$

The successive generations lead to the refinement of the formulas for the error and the standard parameters of both systems, resulting in incremental improvements.

$$\begin{cases} \dot{x} = 35(y_1-x_1)+(x_1-x_2)+y_1z_1 \\ \dot{y} = -y_2-x_1z_1+80x_1 \\ \dot{z} = x_1y_1-z_2-\frac{5}{3}z_1 \end{cases} \tag{21}$$

The construction of a model for both the Qi chaotic system and the target Lü system is achieved through establishing their system equations and mathematical models, accompanied by the plotting of error and state synchronization curves. The purpose of this process is to demonstrate the effectiveness of this approach in synchronizing systems with varying structures.

Here, the initial values of the two systems are: $(x_1, y_1, z_1)=(1,1,1), (x_2, y_2, z_2)=(30,30,30)$.

Fig. 4 and Fig. 5 depict the error and state synchronization diagrams for both the Qi chaotic system and the Lü system, which were created using simulation software.

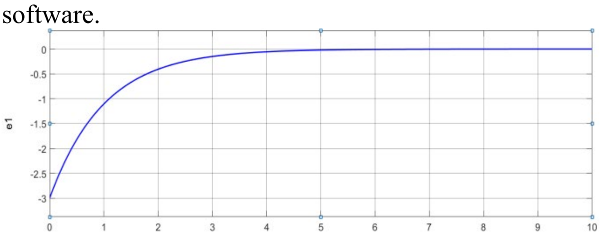
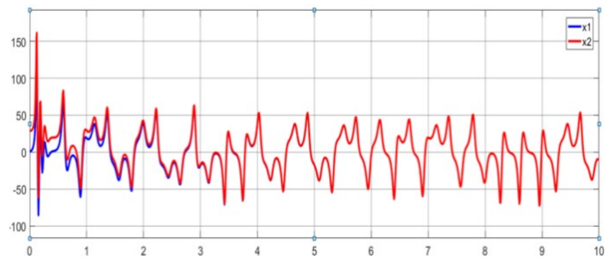


Fig. 4 Simulation error curve of synchronous controller designed by center translation method



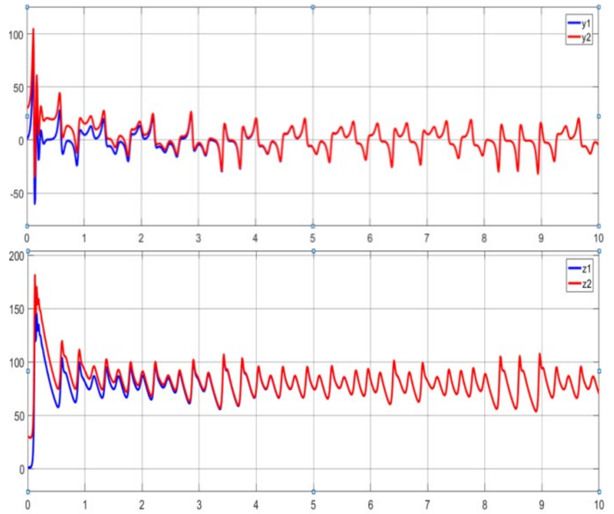


Fig. 5 Simulated state synchronization curve of synchronization controller designed with direct method

After analyzing the simulation data depicted in the diagrams, it is observed that the system can reach its equilibrium state in approximately 4.5 seconds.

4. Conclusion

The direct method for designing synchronous controllers is both theoretically simple and practically efficient, providing significant savings in resources such as time and effort. With its ability to address synchronization problems in systems with different structures, it is widely considered an effective approach.

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